Digital Signal Processing: Exam #1 Review

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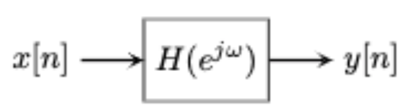
Preamble

This guide serves as a general review of some of the topics for the first exam for the course Digital Signal Processing. Note that there may be typos so use it at your own risk. If you spot a mistake, please let me know so that I can edit it.

1 LSS Review

Linear, Time-Invariant (LTI) Systems

LTI systems are "special" in a way, as the output of an LTI system is somewhat trivial as long as we know the impulse response, generally referred to as $h(t)$. Consider the following block diagram:



If we are given that this system is LTI, it follows the convolution property that $y(t) = x(t) * h(t)$ in continuous-time and $y[n] = x[n] * h[n]$ in discrete-time. For LTI systems, we can take the respective CTFT or DTFT of the input signal to look in the frequency domain that tells us a bit more. Here are a bunch of random facts about LTI systems that I think you might find useful:

- LTI systems cannot create/add new frequencies to input signals.

- If $X(j\Omega) = 0$ for some Ω, then this tells us that $Y(j\Omega) = 0$. That is, if the frequency is not present in the input, then it is not present in the output either.

- LTI systems however can either scale up or scale down the frequencies that are present in an input signal.
  - This follows from the multiplicative property in the frequency domain, (i.e. $Y(j\Omega) = X(j\Omega) \cdot H(j\Omega)$). If $H(j\Omega)$ was a low-pass filter, we can scale down the frequencies from the input. Note that this is also saying that LTI systems can *kill* frequencies.

- LTI systems can change the phase of the input frequencies.
  - This is simply saying $\angle Y(j\Omega) = \angle X(j\Omega) + \angle H(j\Omega)$.

- Complex exponentials (or sinusoids) are eigenfunctions of LTI systems.
  - This is saying that if our input to an LTI system was a complex exponential, then the output would simply be a scaled and (phase) shifted signal according to the frequency reponse, $H(j\Omega)$:

$$y[n] = |H(e^{j\omega_0})| \cdot e^{j(\omega_0 n + \angle H(e^{j\omega_0}))}, \tag{1}$$

    where ω_0 is the input frequency.

DTFT & DTFT Properties

The DTFT and its synthesis are given by the equations

$$X(e^{j\omega}) = \sum_{n=\infty}^{\infty} x[n]e^{-j\omega n} \tag{2}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega. \tag{3}$$

With this, we also have the following important properties:

- **Time Reversal:**

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \tag{4}$$

$$x[-n] \xleftrightarrow{\text{DTFT}} X(e^{-j\omega}) \tag{5}$$

- **Time Shifting:**

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \tag{6}$$

$$x[n-n_0] \xleftrightarrow{\text{DTFT}} e^{-j\omega n_0} \cdot X(e^{j\omega}) \tag{7}$$

- **Frequency Shifting (Modulation) Property:**

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \tag{8}$$

$$e^{j\omega_0 n} \cdot x[n] \xleftrightarrow{\text{DTFT}} X(e^{j(\omega-\omega_0)}) \tag{9}$$

- **Differentiation in Frequency Property:**

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \tag{10}$$

$$n \cdot x[n] \xleftrightarrow{\text{DTFT}} j\frac{d}{d\omega}X(e^{j\omega}) \tag{11}$$

$$\tag{12}$$

- **Multiplication Property:**

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \tag{13}$$

$$h[n] \xleftrightarrow{\text{DTFT}} H(e^{j\omega}) \tag{14}$$

$$x[n] \cdot h[n] \xleftrightarrow{\text{DTFT}} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})H(e^{j(\omega-\theta)}) \tag{15}$$

  Note that equation (12) indicates that the multiplication of two signals in the time domain is the periodic convolution of the two signals in the frequency domain.

- **Convolution Property:**

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \tag{16}$$

$$h[n] \xleftrightarrow{\text{DTFT}} H(e^{j\omega}) \tag{17}$$

$$x[n] * h[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \cdot H(e^{j\omega}) \tag{18}$$

- **Parseval's Theorem:**

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \tag{19}$$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega \tag{20}$$

Miscellaneous

Here are also some facts that I think you might find useful:

- **Even & Odd Signals.** If a signal, say $x(t)$ is even, then $x(t) = x(-t)$. If the signal is odd, then $x(-t) = -x(t)$.

- **Sifting Properties of Impulses.** The "sifting property of impulses" says that integrating a signal multiplied by an impulse returns the value of the function at the location of the impulse:

$$\int_a^b x(t)\delta(t-c)dt = x(c). \tag{21}$$

  Note that this equation implicitly assumes that $a \leq c \leq b$.

- **Geometric Series Formula.** The geometric series formula tells us

$$\sum_{i=0}^{n} ar^n = \frac{a}{1-r}, \tag{22}$$

  where $|r| < 1$.

- **Complex Exponentials.** Taking the DTFT or CTFT of a complex exponential yields an impulse:

$$e^{j\omega_0 n} \xleftrightarrow{\text{DTFT}} 2\pi\delta(\omega - \omega_0) \tag{23}$$

$$e^{j\Omega_0 t} \xleftrightarrow{\text{CTFT}} 2\pi\delta(\Omega - \Omega_0) \tag{24}$$

- **Sinc Function.** The sinc function in the frequency domain is a "box function" with cutoff frequency determined by the sinc. For example the signal

$$x(t) = \frac{3\sin(1000\pi t)}{\pi t} \tag{25}$$

  yields an $X(j\Omega)$ that is a box with gain 3 and cutoff frequency of $|\Omega| \leq 1000\pi$.

- **Periodicity of DTFT Frequencies.** The name suggests it all. We generally consider the fundamental DT frequencies, that is the frequencies within the range $\omega \in [-\pi, \pi]$. Note that this implies that $\omega = 0, \pi$ are the smallest and largest frequencies in DT terms, respectively.

2 Sampling & Aliasing

Sampling

For sampling, we largely deal with the impulse sampled signal, where if our input signal was $x(t)$ then the CTFT of the impulse sampled signal, $x_p(t)$ is

$$\begin{aligned}
x_p(t) &= x(t) \cdot p(t) && (26)\\
X_p(j\Omega) &= \frac{1}{2\pi} X(j\Omega) * P(j\Omega) && (27)\\
X_p(j\Omega) &= \frac{1}{2\pi} X(j\Omega) * \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\Omega - n\Omega_T) && (28)\\
X_p(j\Omega) &= \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j\Omega) * \delta(\Omega - n\Omega_T) && (29)\\
X_p(j\Omega) &= \frac{1}{T} \sum_{n=-\infty}^{\infty} X(j(\Omega - n\Omega_T)). && (30)
\end{aligned}$$

Don't forget the scaling of the $1/T$!!! You should also know that the ideal reconstruction filter takes the mathematical form

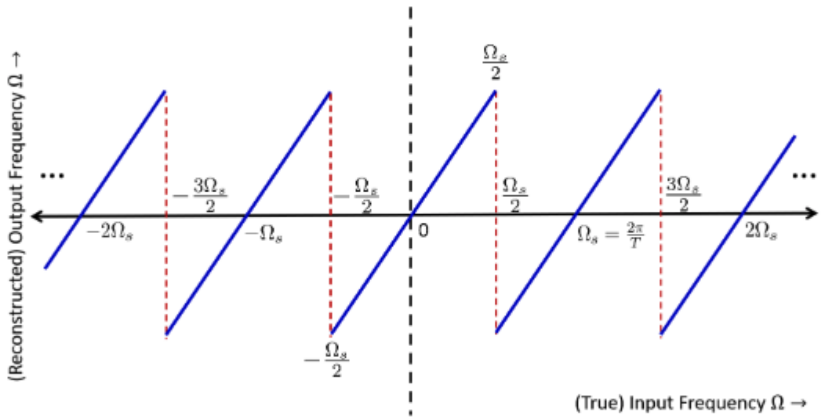
$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \leq \pi/T \\ 0, & \text{otherwise.} \end{cases} \tag{31}$$

Aliasing

Note that if we didn't have aliasing, then the input signal $x(t)$ is equal to the output $y(t)$, the reconstructed signal. That is, our Nyquist criteria was satisfied:

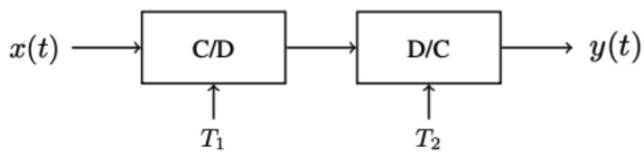
$$\Omega_{\text{s}} \geq 2 \cdot \Omega_{\text{max}}, \tag{32}$$

where Ω_{s} is our sampling frequency and Ω_{max} is the bandwidth of our input signal. Now what if the Nyquist criteria wasn't satisfied? We pretty much have two options: (1) use an anti-aliasing filter to avoid aliasing (in this case, our output would be equivalent to the filtered signal) or (2) use the frequency folding chart and/or plot and find the aliased signal.



The General Procedure

Consider the following block diagram of a digital signal processing system:



The general procedure is the following:

1. Detect if there is aliasing. Some questions might ask you to straight away to determine the output (reconstructed signal) of the system. If there is no aliasing, the reconstructed signal is equal to the input.

2. Plot $X(j\Omega)$ to find the plot of the CTFT of the impulse sampled signal, $X_p(j\Omega)$. You'll find that doing this will make your life easier. Add any overlaps that occur as a result of aliasing.

3. Find the output of the system by going through an ideal reconstruction filter, unless otherwise stated.

3 Quantization

Some notes on quantization:

- ADCs can only detect values between some maximum $n_{\max}$ and some minimum $n_{\min}$. Thus, if you had a signal $g[n]$ where

$$g[n] > n_{\max} \tag{33}$$
$$g[n] < n_{\min}, \tag{34}$$

  then the ADC goes in saturation, which results in clipping.

- We compute the dynamic range of an ADC by

$$R = n_{\max} - n_{\min}. \tag{35}$$

- An ADC is specified by a certain number of bits, say B. It can only store (or generate) 2^B unique values.

- However, note that the signal $g[n]$ can take infinite values, but an ADC can only have $N = 2^B$ values. As a result, we need to map some values to the same value. This is **quantization**.

- More formally, quantization is how an ADC can connect all possible values between some $n_{\max}$ and $n_{\min}$ to 2^B unique numbers, where B is generally pre-specified.

- **Uniform quantization** is equally dividing our range into 2^B pieces and then assigning our discrete signal to the middle of that piece.

  - In order to divide our range evenly (or uniformly), we can divide the range by $N = 2^B$ to get our quantization interval:

$$\delta = \frac{\text{Range}}{2^B} \tag{36}$$

- So how do we actually do the mapping? We have to consider the maximum quantization error:

$$\text{Maximum Quantization Error} = \delta/2 \tag{37}$$

  This maximum quantization error intuitively makes sense because in uniform quantization, we're assigning our signal to the **middle** of the interval. To do the mapping, you should take the quantization value with $\pm\delta/2$ and map it to the corresponding "middle" value.

The following page has some practice problems.

Practice Problems

Problem 1 (LTI Systems)**.** Consider a discrete-time LTI system, whose impulse response is given as

$$h[n] = \begin{cases} -1, & n = 0, \\ 1, & n = 4, \\ 0, & \text{otherwise.} \end{cases} \tag{38}$$

(a) Let $H(e^{j\omega})$ denote the frequency response of the LTI system. Determine $|H(e^{j\frac{\pi}{4}})|$ and $\angle H(e^{j\frac{\pi}{4}})$.

(b) Consider a discrete-time signal $x[n] = \{1, 0, 3, 5\}$ that is the input to this system.

- Compute $y[n]$, the output of this system, by using *only* the convolution property of the DTFT.
- Now suppose that the output of another input signal, say $\tilde{y}[n]$, has the following relationship with $y[n]$:

$$\tilde{Y}(e^{j\omega}) = Y(e^{j\omega})e^{-j3\omega}. \tag{39}$$

  What is this new input signal, $\tilde{x}[n]$?

Problem 1:

(a) $h[n] = -\delta[n] + \delta[n-4]$

$\mathcal{F}\{h[n]\} = H(e^{j\omega}) = 1 + e^{-j4\omega}$.

$H(e^{j\pi/4}) = -1 + e^{-j\pi}$
$= -2 = 2e^{j\pi}$

$|H(e^{j\pi/4})| = 2$.
$\angle H(e^{j\pi/4}) = \pi$.

(b) $F\{x[n]\} = X(e^{j\omega}) = 1 + 3e^{-j2\omega} + 5e^{-j3\omega}$.

$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega}) = (1 + 3e^{-j2\omega} + 5e^{-j3\omega})(-1 + e^{-j4\omega})$

$= -1 - 3e^{-j2\omega} - 5e^{-j3\omega} + e^{-j4\omega} + 3e^{-j6\omega} + 5e^{-j7\omega}$

$y[n] = \{-1, 0, -3, -5, 1, 0, 3, 5\}$.

We know that $\tilde{Y}(e^{j\omega}) = Y(e^{j\omega})e^{-j3\omega}$

$\rightarrow \tilde{Y}(e^{j\omega}) = H(e^{j\omega})\underbrace{X(e^{j\omega})\, e^{-j3\omega}}_{\text{shift in time domain.}}$.

$\tilde{x}[n] = \{1, 0, 3, 5\}$

$n = 3$.

Problem 2 (Sampling)**.** Justify all of your answers.

(a) Consider a continuous-time signal $g(t) = \delta(t-5)$. Can we use the Nyquist sampling theorem to sample this signal?

(b) Consider a continous-time signal $x(t) = \cos(2000\pi t + \pi/3)$ that is sampled using a frequency of 6000π radians/second. Provide a closed-form expression for the discrete-time signal $x[n] = x(nT)$.

Problem 2:

(a) $g(t) = \delta(t-5)$

$G(j\Omega) = e^{-j5\Omega}$.

↳ not bandlimited!

⇒ we cannot use the sampling theorem for sampling $g(t)$.

(b) $x(t) = \cos(2000\pi t + \pi/3)$; $T = 1/3000$ seconds

$x[n] = x(nT) = \cos(2000\pi n (1/3000) + \pi/3)$
$= \cos(2\pi/3 \, n + \pi/3)$.

Problem 3 (2017, Exam 1)**.** Consider the continuous-time signal $g(t) = 4+3\cos(100\pi t-\pi/4) + 4\sin(200\pi t)$. This signal is sampled with a sampling frequency of $f_s = 25$ Hz to obtain $g[n]$.

(a) Provide an expression for the sampled signal $g[n]$.

(b) Suppose $g[n]$ is reconstructed into a signal $\hat{g}_a(t)$ using sinc interpolation. Provide an expression for this signal and justify the reasoning behind your answer.

Problem 3:

(a) $g(t) = 4 + 3\cos(100\pi t - \pi/4) + 4\sin(200\pi t)$

We have $f_s = 25\,Hz$ which yields $T = 1/25$ seconds.

$g(nT) = g[n] = 4 + 3\cos(4\pi n - \pi/4) + 4\sin(8\pi n)$.

Note that $\sin(8\pi n) = 0$.

$\Rightarrow g[n] = 4 + 3\cos(4\pi n - \pi/4)$.

(b) There are several ways to solve this, but one approach can be to "Eulerize" the cosine and look at it directly:

$$\cos(4\pi n - \pi/4) = \frac{e^{j(4\pi n - \pi/4)} + e^{-j(4\pi n - \pi/4)}}{2}$$

$$= \frac{e^{j4\pi n} \cdot e^{-j\pi/4} + e^{-j4\pi n} e^{j\pi/4}}{2}$$

Note that $e^{j4\pi n} = e^{-j4\pi n} = (1)^n$

$$= \frac{e^{-j\pi/4} + e^{j\pi/4}}{2} = \cos(\pi/4).$$

Thus, $\hat{g}_a(t) = 4 + 3\cos(\pi/4)$.

∎

Consider the block diagram of a DSP system shown below in Fig. 3.

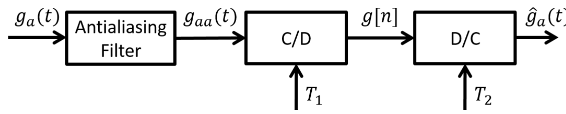


Figure 3: Block diagram of a DSP system.

Further, consider a continuous-time signal $x(t)$ whose continuous-time Fourier transform (CTFT) is in Fig. 4.

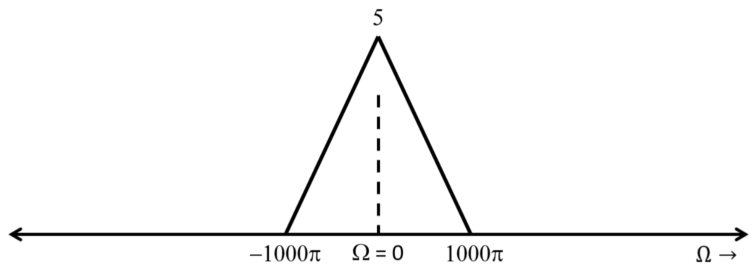


Figure 4: Plot of the CTFT of a continuous-time signal $x(t)$.

The signal $g_a(t)$ being input to the DSP system in Fig. 3 is defined as

$$g_a(t) = x(t)\cos(2000\pi t)\cos(3000\pi t).$$

Further, define $T_1 = \frac{1}{6000}$ seconds and $T_2 = \frac{1}{8000}$ seconds. Finally, suppose the anti-aliasing filter in Fig. 3 has the following frequency response:

$$H_{aa}(j\Omega) = \begin{cases} e^{-j10\Omega}, & |\Omega| \leq 6000\pi, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Provide a labeled plot of the CTFT of $g_a(t)$.

(b) Provide a closed-form expression for $g_{aa}(t)$ as a function of $x(t)$. Justify your answer.